**Chap 1: Complex Numbers & Complex Functions**

1. **Complex Numbers**

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| **Definition** | * A complex number is an ordered pair of real number :   + The real part:   + The imaginary part:   + The imaginary unit: , where   + If is said to be purely imaginary   + If is said to be purely real | |
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| **Modulus & Conjugate Numbers** | * The modulus of a complex number is: * The conjugate number of a complex number is: * Properties of conjugate number: | |
| **Arithmetical Operations** | * **“”:** * **“”:** * **“”:** communicative and distributive laws   + When * **“”:**   + Division of a complex number by a real number   + Division of 2 complex numbers | |
| **Triangle Inequality** | * We have: * If and , then we have: | |

1. **Polar Form – Exponential Form of Complex Numbers**
2. **Polar Form**

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| **Complex Plane** | * If , then the geometrical representation of a complex number as a point in the plane is:      * The horizontal x-axis is the **real-axis** * The vertical y-axis is the **imaginary-axis** * The conjugate number is the reflection of across the real-axis * The length of the vector is: * The distance between is: | | |
| Example |  |  |  |

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| **Polar Form** | * We can present the complex point by polar coordinates : * The argument of is: * The principle value of is: |
| Example | Change 🡪 polar form?   * So,   Change 🡪 polar form?   * So, |
| **Operations** | * **“”:** * **“”:** * **Engineers’ j-operator**: if , then: |
| Example | Find the product of and in polar form?   * and   Find the division of and ? |

1. **Exponential Form**

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| **Exponential Form** | * The complex exponential function of is: * The polar form of can be written in (Euler Formula) exponential form: * Some properties of exponential form: |
| Example | Change 🡪 rectangular form?  Change 🡪 rectangular form?  Change 🡪 rectangular form?  Change 🡪 exponential form?   * We have: * The polar form is: * The exponential form is: |

1. **Power – Root of Complex Numbers**
2. **Power of Complex Numbers**

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| **Power of Complex Numbers** | * (De Moivre’s theorem) If is a positive integer, then: |
| Example | Express 🡪 rectangular form?  Express 🡪 rectangular form? |

1. **Root of Complex Numbers**

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| **Root of Complex Numbers** | * If is a positive integer, then has distinct values: * The principal nth root of : |
| Example | Find the roots of ?   * Therefore, * Another way,   Find the roots of ?   * Therefore: |

1. **Complex Function**

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| **Definition** | * A complex function defined on a domain D is a rule that assigns a complex number to every   + is the complex variable   + is the value of   + D is the domain |
| Example | Let at . Find and ?   * We know that , so: * We can find: * Therefore,   Let at . Find and ?   * We have: * So: * We can find: |
| **Limit and Continuity** | * The limit of is: * Properties of 2 limits:      * A function is continuous at if: * If is continuous at all points of its domain ⇒ is a continuous function |
| **Derivative** | * The derivative of at is defined by   then is said to be differentiable at   * The differentiation rules are the same as in real calculus * The same chain rule as in real calculus * The same rules for 2 analytic function: |
| Example |  |

1. **Cauchy-Riemann Equation – Laplace Equation**

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| **Cauchy-Riemann Equation** | * The Cauchy- Riemann equations in rectangular: * The Cauchy- Riemann equations in polar form:   **The application of Cauchy-Riemann Equation**   * If:   + The partial derivatives exist and continue in the domain D   + The Cauchy-Riemann equations are satisﬁed * Then:   + is **diﬀerentiable** at and   + is an **analytic function**     - Polynomial is analytic everywhere     - Rational functions are analytic on their domains     - In a region , an **isolated singularity**  is where isn’t analytic |
| Example | Show that is diﬀerentiable. Find ?   * The partial derivatives:   (exist and continuous everywhere)  (exist and continuous everywhere)   * Therefore is diﬀerentiable     Show that is diﬀerentiable. Find ?   * The partial derivatives:   (exist and continuous everywhere)  (exist and continuous everywhere)   * Therefore is diﬀerentiable |
| Show that is an analytic function?   * The partial derivatives:   (exist and continuous everywhere)  (exist and continuous everywhere)   * Therefore is analytic for all   Show that is an analytic function?   * exists when * Since is a rational function. Therefore is analytic for all excepts |
| **Laplace Equation** | * If is analytic in domain D, then both and satisfy the Laplace equations: * If satisfy the Laplace equation & Cauchy-Riemann equation:   + and are **harmonic** in D   + is **harmonic conjugate function** of in D |
| Example | Show that is harmonic. Find the harmonic conjugate function?   * We have: * Apply the Laplace Equation: * Therefore, is a harmonic function * Apply the Cauchy-Riemann Equation: * Integrating with respect to we have: * Differentiating with respect to we have: * From (1) and (2): * Therefore: |

1. **Elementary Complex Functions**

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| **Exponential Functions** | * The exponential function: * Property:  |  | | --- | | 1. & | | 1. is analytic everywhere and | | 1. The periodic properties: | |
| **Operation** | * With positive number: |
| Example | Let 🡪 ?   * We have: * Therefore, |
| **Trigonometric & Hyperbolic Functions** | * If then we have the trigonometric function of * Furthermore, we have: |